

SPECIALISATIONS OF ALGEBRAICALLY CLOSED VALUED FIELDS

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ABSTRACT. For an algebraically closed valued field K and its residue field k we will consider the residue map $res : K \rightarrow k$ as a (partial) specialisation, and show that given $tr.d(K/k) = \kappa$ for an infinite κ , res is a κ -universal specialisation.

INTRODUCTION

We will consider the structure (K, k, res) where K is an algebraically closed valued field, k its residue field, and $res : K \rightarrow k$ the residue map (considered as a partial function) from the point of view of Zariski structures and their specialisations, see [10, Example 2.2.5 and 2.2.6] for concrete examples. Algebraically closed valued fields are very well studied in various model theoretic settings [2, 3]. The main difference of this setting is that we will consider k as an elementary substructure of K ; and res will be identity on k . The first order theory of specialisations are recently considered by Onshuus and Zilber; they also observed that this theory is a generalisation of ACVF. In this work they are particularly interested in a special class of specialisations called κ -universal specialisations[5].

In section 1 we will briefly recall well known and fundamental facts about Zariski structures and their specialisations; much more on them can be found in [10]. In section 2 we will show that one can consider the first order theory ACVF of algebraically closed valued fields in the setting of specialisations, and show that res is a κ -universal specialisation if $tr.d(K/k) = \kappa$. Both of these facts are already very well known to model theorists. However, up to this point they were not pinned down rigorously. In this short note, we write these observations rigorously.

1. PRELIMINARIES

1.1. Zariski Structures. Let \mathcal{L} be a language and M be an \mathcal{L} structure. The closed subsets of M^n for all $n \in \mathbb{N}$ are defined as follows: The subsets defined by primitive relations of \mathcal{L} are (definably) *closed*, and moreover the sets which are given by positive quantifier free \mathcal{L} -formulas are *closed*. Equivalently they can be defined by the following topological axioms:

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- (1) Arbitrary intersection of closed sets is closed;
- (2) Finite unions of closed sets are closed;
- (3) M is closed;
- (4) The graph of equality is closed;
- (5) Any singleton in M is closed;
- (6) Cartesian products of closed sets are closed;
- (7) The image of a closed set under a permutation of coordinates is closed;
- (8) For $a \in M^k$ and closed $S \subseteq M^{k+l}$, the set $S(a, M^l)$ is closed.

A *constructible set* is defined to be a boolean combination of closed sets. In other words they are nothing but sets defined by quantifier free \mathcal{L} -formulas. A set S is said to be *irreducible* if there are no proper relatively closed subsets $S_1, S_2 \subset S$ such that $S = S_1 \cup S_2$. A topological structure is called *Noetherian* if it satisfies the descending chain condition (DCC) for closed sets.

Remark that in Noetherian topological structures any closed set S can be written as $S = S_1 \cup \dots \cup S_k$ where $S_1, \dots, S_k \subset S$ are distinct and relatively closed; moreover they are unique up to ordering. They are called *irreducible components* of S .

A topological structure is said to be semi-proper if it satisfies the following condition:

(SP): Semi Properness: For a closed irreducible $S \subset M^n$ and a projection $\text{pr} : M^n \rightarrow M^m$, there is a proper closed subset $F \subset \overline{\text{pr}(S)}$ such that $\overline{\text{pr}(S)} \setminus F \subset \text{pr}(S)$

The *dimension*, denoted by \dim is a function from definable sets of Noetherian topological structure to natural numbers which satisfies:

- (DP):** Dimension of a Point: $\dim(a) = 0$ for all $a \in M$;
- (DU):** Dimension of Unions: $\dim(S_1 \cup S_2) = \max(\dim(S_1), \dim(S_2))$ for closed S_1 , and S_2 ;
- (SI):** Strong irreducibility: For any irreducible $S \subseteq_{cl} U \subseteq_{op} M^n$ and any closed $S_1 \subsetneq S$, $\dim(S_1) < \dim(S)$;
- (AF):** Addition Formula: For any irreducible closed $S \subseteq_{cl} U \subseteq_{op} M^n$ and a projection $\text{pr} : M^n \rightarrow M^m$,

$$\dim(S) = \dim(\text{pr}(S)) + \min_{a \in \text{pr}(S)} (\text{pr}^{-1}(a) \cap S)$$

- (FC):** Fiber Condition: Given $S \subseteq_{cl} U \subseteq_{op} M^n$ and a projection $\text{pr} : M^n \rightarrow M^m$, there is a relatively open $V \subseteq_{op} \text{pr}(S)$ such that, for any $v \in V$

$$\min_{a \in \text{pr}(S)} (\dim(\text{pr}^{-1}(a) \cap S)) = \dim(\text{pr}^{-1}(v) \cap S)$$

A Noetherian, semi-proper topological structure together with a dimension function which satisfies the above conditions is said to be a *Noetherian Zariski Structure*.

A (Noetherian) Zariski structure is said to be *complete* if it satisfies:

(P): Properness: projections of closed sets are closed. I.e. For any closed $S \subseteq M$, $pr_{i_1, \dots, i_m}(S)$ is closed.

It is called *quasi-compact* if:

(QC): Quasi-compactness: For a finitely consistent family \mathcal{C} of closed subsets of M^n , $\cap \mathcal{C}$ is not empty.

Example 1.1.

An algebraically closed field K in language \mathcal{L}_{zar} for algebraic varieties (which only has predicates for Zariski closed subsets of K^n for $n \in \mathbb{N}$) together with the Krull dimension is a Zariski structure.

Furthermore, let $\mathbb{P}^n K$ be the projective n -space over K (all $n \in \mathbb{N}$) in the language for projective varieties (which only has predicates for projective Zariski closed subsets of $\mathbb{P}^n K$ for $n \in \mathbb{N}$), together with the Krull dimension is a Zariski structure.

Details for both cases can be found in [7, chap. 1-4].

Fact 1.2. A Zariski Structure M has quantifier elimination. In more geometric words, every definable set is constructible.

1.2. Specialisations.

Definition 1.3. Let π be a (partial) function $\pi : M \rightarrow M_0$ where M_0 is a Noetherian Zariski Structure, M an elementary extension of M_0 and for every formula $S(\bar{x})$ over \emptyset , defining an M -closed set and for every $\bar{a} \in M^n$ (which is also in the domain of π), $M \models S(\bar{a})$ implies $M_0 \models S(\pi\bar{a})$. Such a function is said to be a specialisation. We will also call the tuple (M, π) a specialisation.

A specialisation is said to be κ -universal if, given any $M' \succ M \succ M_0$, any $A \subseteq M'$ with $|A| < \kappa$ and a specialisation $\pi_A : M \cup A \rightarrow M_0$ extending π , there is an embedding $\sigma : A \rightarrow M$ over $M \cap A$ such that $\pi_A|_A = \pi \circ \sigma$.

An ω -universal specialisation is said to be universal.

Remark 1.4. Let (M, π) be a κ -universal specialisation, then M is κ -saturated. Indeed, let $p(x) \in S_1(B)$, for some $B \subseteq M$ with $|B| < \kappa$. Suppose a realises $p(x)$ in some elementary extension M' of M , and put $A = B \cup \{a\}$. Hence by κ -universality of π , there is an embedding $\sigma : A \rightarrow M$ over B such that $\sigma(a)$ realises $p(x)$ in M .

The following is one of the natural examples of specialisation; and it is main object we will work with in this paper.

Example 1.5. An algebraically closed valued field K in the language \mathcal{L}_{zar} for algebraic varieties together with the residue map $res : K \rightarrow k$ where k is the residue field is a specialisation.

However, note that res is a partial specialisation in this example. One can easily give an example of a total specialisation, the following is Example 2.2.4 in [10].

Example 1.6. Let ${}^*\mathbb{R}$ be the hyperreal numbers. Then the standard part map from ${}^*\mathbb{R}$ to \mathbb{R} (which is a partial map) induces a total specialisation $\pi : \mathbb{P}^1 {}^*\mathbb{R} \rightarrow \mathbb{P}^1 \mathbb{R}$. Note that ${}^*\mathbb{R}$ interprets an elementary extension ${}^*\mathbb{C}$ of \mathbb{C} , therefore one can consider $\pi : \mathbb{P}^1 {}^*\mathbb{C} \rightarrow \mathbb{P}^1 \mathbb{C}$ in the same way, and show that it is a total specialisation.

The following facts are Proposition 2.2.7 and 2.2.15 in [10]. The first one is due to more general results of Weglorz [9].

Fact 1.7. Let M be a quasi-compact, and complete Zariski structure, and $M' \succeq M$. Then there is a total specialisation $\pi : M' \rightarrow M$. Moreover, any (partial) specialisation can be extended to a total specialisation.

Fact 1.8. Let M be a Zariski structure, and $M' \succeq M$. Then there is a κ -universal specialisation (M', π) . Moreover, if M is quasi-compact, then π is a total specialisation.

Another important tool related to specialisations that we will use in the proof of the main result is the infinitesimal neighbourhood of a point.

Definition 1.9. An infinitesimal neighbourhood of $a \in M^n$ is the set $\mathcal{V}_a = \pi^{-1}(a)$.

2. SPECIALISATIONS OF ALGEBRAICALLY CLOSED VALUED FIELDS

In this chapter we will consider the structure (K, k, res) where K is an algebraically closed valued field, k its residue field and $res : K \rightarrow k$ is the residue map as a specialisation. For this consideration we are forced to make some restrictions. As we need to have $k \prec K$, we restrict ourselves to the case $\text{Char}(K) = \text{Char}(k)$. Given that we are in the equi-characteristic, both K and k are algebraically closed fields over the same prime subfield. Then one can (uncanonically) embed k into K , also one can choose this embedding in such a way that res is identity on k (by replacing the isomorphic copy of k in K with k if necessary). But then by quantifier elimination (Fact 1.2) this embedding is elementary, and we have $k \prec K$.

We will work with the relational language \mathcal{L}_{Zar} , which only has predicates for varieties in k^n (all $n \in \mathbb{N}$), whose relations are Zariski closed subsets of k^n (all $n \in \mathbb{N}$), and with the expanded language $\mathcal{L}^{res} := \mathcal{L}_{\text{Zar}} \cup \{res\}$ where res will be interpreted as the residue map. Note that k is \mathcal{L}^{res} definable in K by the formula $\exists x(res(x) = y)$. Hence we will write our structures as (K, res) . From now on K will always be an algebraically closed valued field, k the (definable) residue field, $res : K \rightarrow k$ the residue map, \mathcal{O} the corresponding valuation ring, and \mathcal{M} the valuation ideal.

Recall that ACVF is the complete theory given by the following sentences:

- (1) K is algebraically closed.
- (2) There are $x, y \in K^\times$ such that $v(x) < v(y)$, where v is the valuation on K .
- (3) Characteristic of k and K .

in various languages due to Robinson's analysis of algebraically closed valued fields [4, 6].

It is easy to see that one can write sentences 1 and 3 in the language \mathcal{L}^{res} , one can also write 2 but it takes a bit more effort. First observe that \mathcal{O} is \mathcal{L}^{res} definable, by the formula $\exists y(res(x) = y)$. Then define $Res : K \times K \rightarrow k$ as follows:

$$Res(x, y) = \begin{cases} res(xy^{-1}) & \text{if } v(x) \geq v(y) \\ 0 & \text{otherwise} \end{cases}$$

So Res is \mathcal{L}^{res} definable by the formula

$$\psi(x, y, z) : (xy^{-1} \in \mathcal{O} \wedge z = res(xy^{-1})) \vee (xy^{-1} \notin \mathcal{O} \wedge z = 0)$$

Now sentence 2 can be written as $\exists x \exists y (Res(x, y) = 0)$. Therefore, one can indeed consider the theory ACVF to be living inside the theory specialisations (with some restrictions).

We start with a straightforward adaptation of a standard fact.

Lemma 2.1. *Let K be an algebraically closed (valued) field in the language \mathcal{L}_{zar} . If $tr.d(K) = \kappa$ then K is κ -saturated. Conversely, if K is κ -saturated, then K has infinite transcendence degree.*

Proof. Let $A \subseteq K$ with $|A| < \kappa$. Let F be the field generated by A . Consider the algebraic closure \overline{F} of F . Observe that $|\overline{F}| < \kappa$. Let $p(x) \in S_1(\overline{F})$. We will look at the positive part $p^+(x)$ of $p(x)$. Notice that $p^+(x)$ is just a family $\{V(I_j)\}_{j \in J}$ of varieties over \overline{F} for ideals $I_j \triangleleft \overline{F}[X]$.

Now consider the descending chain

$$V(I_1) \supseteq V(I_1) \cap V(I_2) \supseteq V(I_1) \cap V(I_2) \cap V(I_3) \supseteq \dots$$

By Noetherianity this chain stabilises at some point; say

$$V = \bigcap_{j=1}^n V(I_j) = V(I_1 + \dots + I_n)$$

Now if $I_1 + \dots + I_n = 0$ then $p(x)$ says that x is transcendental over \overline{F} . Since $tr.d(K) = \kappa$ and $|\overline{F}| < \kappa$ there is an $a \in K$ satisfying $p(x)$. So, suppose $I_1 + \dots + I_n \neq 0$. Then since $\overline{F}[X]$ is a PID, $I_1 + \dots + I_n = (f(X))$ for some $f(X) \in \overline{F}[X]$. Now choose a root a of $f(X)$ such that a satisfies $p(x)$.

For the converse, let $A \subset K$ with $|A| < \kappa$. Consider the field F generated by A . Let $p(x) \in S_1(F)$ be the 1-type saying that x is transcendental over F . By κ -saturation $p(x)$ is realised in K . Then by induction one can show that the type $p(x_1, \dots, x_n) \in S_n(F)$ which is saying x_1, \dots, x_n are algebraically independent over F is realised for all $n \in \mathbb{N}$. Hence transcendence degree of K needs to be infinite. \square

Remark 2.2. *Recall that Infinitesimal neighbourhood of any element b in K is $\mathcal{V}_b = res^{-1}(b)$. Note that in this setting \mathcal{V}_0 is just the valuation ideal \mathcal{M} . Hence for any $b \in k$, the infinitesimal neighbourhood \mathcal{V}_b is just the coset*

$b + \mathcal{M}$. Moreover, by adding ∞ to k in the usual way, one can also consider \mathcal{V}_∞ as a coset.

Theorem 2.3. *Given a specialisation $\pi : K \rightarrow k$, it can be extended to a residue map, $\text{res} : K \rightarrow k'$ of a valuation ring with $k' \succ k$.*

Proof. Let $R = \text{Dom}(\pi)$. Clearly π is a surjective ring homomorphism with $\text{Ker}(\pi) = \mathcal{V}_0$. Hence $\mathcal{V}_0 \triangleleft R$ is maximal, and in particular prime. Then by Chevalley's Place Extension Theorem, there is a valuation ring \mathcal{O} of K such that

$$R \subseteq \mathcal{O} \text{ and } \mathcal{M} \cap R = \mathcal{V}_0$$

where $\mathcal{M} \triangleleft \mathcal{O}$ is the maximal ideal.

Now, consider $f : k \rightarrow k'$ defined by $f(a + \mathcal{V}_0) = a + \mathcal{M}$. This clearly is a ring monomorphism. Furthermore, by quantifier elimination (Fact 1.2) it is an elementary embedding. So assume that we identified the copy of k in k' with k itself.

Next we will show that res extends π . Indeed, if $a \in \mathcal{V}_0 = \mathcal{M} \cap R$, then $\text{res}(a) = 0 = \pi(a)$ as we identified k with its copy in k' . If $a \in \mathcal{V}_b$ for $b \in k^\times$, then $a = b + a_1$ for some $a_1 \in \mathcal{V}_0$. Hence

$$b = \pi(a) = \pi(b + a_1) = b + \pi(a_1) = b + \text{res}(a_1) = \text{res}(b + a_1) = \text{res}(a)$$

□

Therefore whenever an arbitrary specialisation of an algebraically closed valued field is given we can consider its extension to a certain residue map (of a valuation). From now on we will only consider residue map as a specialisation and show that if K has enough transience degree over the residue field k , then residue map has enough universality as a specialisation.

Before we prove the main result, we need to establish the fact that infinitesimal neighbourhoods (i.e. cosets of \mathcal{M}) are large sets in this setting. The following lemma essentially guarantees this fact.

Lemma 2.4. *$K = \text{acl}(k(\mathcal{V}_b))$ for any $b \in k$.*

Proof. Let $S = \{x_i : i < \kappa\}$ be a transcendence basis of K over k . We will show that one can choose x_i such that $v(x_i) > 0$. First observe that if $v(x_i) < 0$, then $v(x_i^{-1}) \geq 0$, and clearly by replacing x_i with x_i^{-1} when necessary we again get a transcendence basis.

Next, if $v(x_i) = 0$, then $x_i = b + m_i$ for some $b \in K^\times$ and $m_i \in \mathcal{M}$. Then $v(x_i) = v(b + m_i) = v(b)$. Clearly m_i is transcendental over k . Moreover, it is also clear that m_i is algebraically independent from x_j for $j \neq i$. Hence \mathcal{V}_0 contains a transcendence basis of K over k ; say $\{m_i : i < \kappa\}$.

By the same argument, it follows that for $b \in k^\times$, $\{b + m_i : i < \kappa\} \subset \mathcal{V}_b = b + \mathcal{M}$ is a transcendence basis. □

Theorem 2.5. *Consider the specialisation (K, res) . If $\text{tr.d}(K/k) = \kappa$ for some $\kappa \geq \aleph_0$, then $\text{res} : K \rightarrow k$ is a κ -universal specialisation.*

Proof. Let $K' \succ K$ and $A \subset K'$ with $|A| < \kappa$ and $res' : K \cup A \rightarrow k$ an extension of res . Look at the field generated by A , call it F . Let $F_0 = K \cap F$.

Let $b_1 \in F$. Let $b = res'(b_1)$, we will show that there is an $a \in K$ such that $tp(a/F_0) = tp(b_1/F_0)$ and $res(a) = b$. If $b_1 \in F_0$ then take $a = b_1$. Suppose $b_1 \in F \setminus F_0$. Then in particular b_1 is transcendental over F_0 , let $F_1 = F_0(b_1)$. So we need to find an element in \mathcal{V}_b which is transcendental over F_0 . But by Lemma 2.4 $|\mathcal{V}_b| \geq \kappa$; and $|F_0| < \kappa \leq \kappa$. Hence $\mathcal{V}_b \not\subseteq acl(F_0)$. Choose $a \in \mathcal{V}_b \setminus acl(F_0)$. Then by mapping b_1 to a one can find an embedding $\sigma_1 : F_1 \rightarrow K$ over F_0 such that $res' = (res \circ \sigma_1)|_{F_0}$.

By the same construction, one can inductively extend σ_1 to an embedding $\sigma : F \rightarrow K$ over F_0 such that $res' = (res \circ \sigma)|_{F_0}$. \square

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